Wzór zwarty na ciąg Fibonacciego

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1 Metoda równania charakterystycznego

Wyprowadzenie wzoru "zwartego" na n-tą liczbę ciągu Fibonacciego:

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_0 = 0 \\ F_1 = 1 \end{cases}$$

$x^2 = x + 1$
$x^2 - x - 1 = 0$
$\Delta = \sqrt{5}$
$x_1 = \frac{1+\sqrt{5}}{2}$
$x_2 = \frac{1-\sqrt{5}}{2}$

$F_n = A * (\frac{1+\sqrt{5}}{2})^n + B * (\frac{1-\sqrt{5}}{2})^n$
$F_0 = 0 = A + B \rightarrow B = -A$
$F_1 = 1 = \frac{A+A\sqrt{5}+B-B\sqrt{5}}{2}$
$2 = A + A\sqrt{5} + B - B\sqrt{5}$
$2 = 2A\sqrt{5}$
$A = \frac{\sqrt{5}}{5}$
$B = -\frac{\sqrt{5}}{5}$

Rozwiązanie:

$$F_n = \frac{\sqrt{5}}{5}(\frac{1+\sqrt{5}}{2})^n - \frac{\sqrt{5}}{5}(\frac{1-\sqrt{5}}{2})^n$$

2 Funkcja tworząca

$$\begin{cases} F_n = F_{n-1} + F_{n-2} \\ F_0 = 0 \\ F_1 = 1 \end{cases}$$

$$F(x) = \sum_{n\geq 0} F_n x^n = F_0 + F_1 x + \sum_{n\geq 2} F_n x^n = F_0 + F_1 x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n$$

$$F(x) = F_0 + F_1 x + \sum_{n\geq 1} F_n x^{n+1} + \sum_{n\geq 0} F_n x^{n+2}$$

$$F(x) = 0 + x + x(F(x) - F_0) + x^2F(x)$$

$$F(x) = x + xF(x) + x^2F(x)$$

$$F(x)(1 - x - x^2) = x$$

$$F(x) = \frac{x}{1 - x - x^2}$$

$$\lambda_1 = \frac{1 - \sqrt{5}}{2}$$

$$\lambda_2 = \frac{1 + \sqrt{5}}{2}$$

$$F(x) = \frac{x}{(1 - \lambda_1 x)(1 - \lambda_2 x)} = \frac{\sqrt{5}}{5} * (\frac{1}{1 - \lambda_1 x} - \frac{1}{1 - \lambda_2 x})$$

$$F(x) = \frac{\sqrt{5}}{5} \sum_{n \geq 0} (\lambda_1^n - \lambda_2^n) x^n = \sum_{n \geq 0} \frac{\sqrt{5}}{5} (\lambda_1^n - \lambda_2^n) x^n$$

Rozwiązanie:

$$F_n = \frac{\sqrt{5}}{5} (\lambda_1^n - \lambda_2^n)$$